1. INTRODUCTION

Aperture antennas are most commonly used at microwave and the millimeter-wave frequencies. There are a large number of antenna types for which the radiated electromagnetic fields can be considered to emanate from a physical aperture. Antennas that fall into this category include reflector antennas, lenses, and horn antennas. The geometry of the aperture may be square, rectangular, circular, elliptical, or virtually any other shape. The term “aperture antenna” usually has more to do with the method used to analyze the antenna than the actual form of the antenna. For instance, array antennas viewed as a continuous field function bounded by a conducting screen can be considered to be an aperture antenna. This would encompass so-called planar (flat-plate) waveguide and microstrip arrays.

Aperture antennas are very popular for aerospace applications because they can be flush-mounted onto the spacecraft or aircraft surface. Their aperture opening can be covered with an electromagnetic (dielectric) window material that is transparent to the RF energy to protect the antenna from the environmental conditions [1]. This is known as a radome, and it is implemented so as not to disturb the aerodynamic profile of the vehicle, which is of special importance to high-speed aircraft or missiles.

In order to evaluate the distant (far-field) radiation patterns, it is necessary to know the surface currents that flow on the radiating surfaces of the antenna aperture. In many instances, these current distributions may not be known exactly and only approximate or experimental measurements can provide estimates. A technique based on the equivalence principle allows one to make reasonable approximations to the electromagnetic fields on, or in the vicinity of, the physical antenna aperture structure, which can then be used to compute far field antenna radiation patterns.

Field equivalence, first introduced by Schelkunoff [2], is a principle by which the actual sources on an antenna aperture are replaced by equivalent sources on an external closed surface that is physically outside the antenna aperture. The fictitious sources are said to be equivalent within a region because they produce the same fields within that region. Another key concept is Huygens’ principle [3], which states that the equivalent source at each point on the external surface is a source of a spherical wave. The secondary waveform can be constructed as the envelope of these secondary spherical waves [4].

Using these principles, the electrical and/or magnetic fields in the equivalent aperture region can be determined with these straightforward, but approximate, methods. The fields elsewhere are assumed to be zero. In most applications, the closed surface is selected so that most of it coincides with the conducting parts of the physical antenna aperture structure. This is preferred because the
disappearance of the tangential electrical components over the conducting parts of the surface reduces the physical limits of integration. The formula to compute the fields radiated by the equivalent sources is exact, but it requires integration over the closed surface. The degree of accuracy depends on the knowledge of the tangential components of the electromagnetic fields over the closed surface.

Aperture techniques are especially useful for parabolic reflector antennas, where the aperture plane can be defined immediately in front of the reflector. Parabolic reflectors are usually electrically large. More surprisingly, aperture techniques can also be successfully applied to small-aperture waveguide horn antennas. However, for very small horn antennas with an aperture dimension of less than approximately one wavelength, the assumption of zero fields outside the aperture fails unless the horn is completely surrounded by a planar conducting flange [5]. In this section, the mathematical formulas will be developed to analyze the radiation characteristics of aperture antennas. Emphasis will be given to the rectangular and circular configurations because they are the most commonly used geometries. Because of mathematical complexities, the results will be restricted to the far-field region.

One of the most useful concepts is the far-field radiation pattern that can be obtained as a Fourier transform of the field distribution over the equivalent aperture, and vice versa. Fourier transform theory is extremely important to the analysis and synthesis of aperture antennas. Obtaining analytical solutions for many simple aperture distributions in order to design aperture antennas is useful. More complex aperture distributions, which do not lend themselves to analytical solutions, can be solved numerically. The increased capabilities of the personal computer (PC) have resulted in its acceptance as a conventional tool for the antenna designers. The Fourier transform integral is generally well behaved and does not present any fundamental computational problems.

Considering the use of the Fourier transform, first consider rectangular apertures in which one aperture dimension is large in wavelengths and the other is small in terms of wavelengths. This type of aperture is approximated as a line source and is treated with a one-dimensional Fourier transform [6]. For many kinds of rectangular aperture antennas such as horns, the aperture distributions in the two principal-plane dimensions are independent. These types of distributions are said to be separable. The total radiation pattern is the product of the pattern functions obtained from the one-dimensional Fourier transforms, which corresponds to the two principal-plane distributions.

If the rectangular aperture distribution cannot be separated, the directivity pattern is found in a similar manner to the line-source distribution, except that the aperture field is integrated over two dimensions rather than one dimension [7]. This double Fourier transform can also be applied to circular apertures.

For all aperture distributions, the following observations are made [8]:

1. A uniform amplitude distribution yields the maximum directivity (where nonuniform edge-enhanced distributions for supergain are considered impractical), but at high sidelobe levels.
2. Tapering the amplitude at the center, from a maximum to a smaller value at the edges, will reduce the sidelobe levels compared with the uniform illumination, but it results in a larger (mainlobe) beamwidth and less directivity.
3. An inverse-taper distribution (amplitude depression at the center) results in a smaller (mainlobe) beamwidth but increases the sidelobe level and reduces the directivity when compared with the uniform illuminated case.
4. Depending on the aperture size in wavelengths and the phase errors, there is a frequency (or wavelength) for which the gain peaks, falling to smaller values as the frequency is either raised or lowered.

Finally, we consider aperture efficiencies. The aperture efficiency is defined as the ratio of the effective aperture area to the physical aperture area. The beam efficiency is defined as the ratio of the power in the mainlobe to the total radiated power. The maximum aperture efficiency occurs for a uniform aperture distribution, but maximum beam efficiency occurs for a highly tapered distribution. The aperture phase errors are the primary limitation of the efficiency of an antenna.

2. HUYGENS’ PRINCIPLE

The principle proposed by Christian Huygens (1629–1695) is of fundamental importance to the development of wave theory [3]. Huygens’ principle states that “Each point on a primary wavefront serves as the source of spherical secondary wavelets that advance with a speed and frequency equal to those of the primary wave. The primary wavefront at some time later is the envelope of all these wavelets” [9]. This is illustrated in Fig. 1 for spherical and
plane waves modeled as a construction of Huygens’ secondary waves. Actually, the intensities of the secondary spherical wavelets are not uniform in all directions but vary continuously from a maximum in the direction of wave propagation to a minimum of zero in the backward direction. As a result, there is no backward-propagating wavefront. The Huygens source approximation is based on the assumption that the magnetic and electric fields are related as a plane wave in the aperture region.

The situation shown in Fig. 2, shows an infinite electromagnetic plane wave incident on an infinite flat sheet that is opaque to the waves. This sheet has an opening that is very small in terms of wavelengths. Accordingly, the outgoing wave corresponds to a spherical wavefront propagating from a point source. That is, when an incoming wave comes against a barrier with a small opening, all except one of the effective Huygens point sources are blocked, and the energy coming through the opening behaves as a single point source. In addition, the outgoing wave emerges in all directions, instead of just passing straight through the slit.

On the other hand, consider an infinite plane electromagnetic wave incident on an infinite opaque sheet shown in Fig. 3 that has a larger opening of dimension a. The field everywhere to the right of the sheet is the result of the section of the wave that passes through this opening. If a is large in terms of wavelengths, the field distribution across the slot is assumed, as a first approximation, to be uniform. The total electromagnetic field at any point to the right of the opening is obtained by integrating the contributions from an array of Huygens sources distributed over the length a. We calculate the electric field at point P on a reference plane located at a distance $R_o$ from this opening by evaluating the integration of these Huygens sources [10]:

$$E = \int E_0 e^{-jkr} \, dy$$

(1)

For points near to the array, the integral does not simplify but can be reduced to the form of Fresnel integrals.

The actual evaluation of this integral is best achieved on a PC computer, which reduces the integral to a summation of N Huygens sources

$$E = \sum_{i=1}^{N} \frac{e^{-jkr_i}}{r_i}$$

(2)

where $r_i$ is the distance from the i-th source to point P. The field variation near the slot opening that is obtained in this way is commonly called a Fresnel diffraction pattern [4].

For example, let us consider the case in which the slot length $a$ is 5 cm and the wavelength is 1.5 cm (corresponding to 20 GHz.) We can use Eq. (2) to compute the field along a straight line parallel to the slot. The field variation for $R_o = 2.5$ cm shown in Fig. 4. For this case, $R_o$ is well within the near field (the so-called Fresnel region.) As we continue to increase $R_o$, the shape of the field variation along this line continues to vary with increasing $R_o$ until we reach the far-field or Fraunhofer region (see the trends in Figs. 4b–4d). Once we have entered the Fraunhofer region, the pattern becomes invariant to range $R$.

For the distance to be in the far field, the following relationship must be met

$$R_o \geq \frac{2a^2}{\lambda}$$

(3)

where $a$ is the width of the slot and $\lambda$ is the wavelength. Thus, the larger the aperture or the shorter the wavelength, the greater will be the distance at which the pattern must be measured if we wish to avoid the effects of Fresnel diffraction.

Huygens’ principle is not without limitations as it neglects the vector nature of the electromagnetic field space. It also neglects the effect of the currents that flow at the slot edges. However, if the aperture is sufficiently large and we consider only directions roughly normal to the aperture, the scalar theory of Huygens’ principle gives very satisfactory results.

Geometric optics (GO) techniques are commonly applied in reflector antennas to establish the fields in the reflector aperture plane. This procedure, referred to as
The aperture field method, is employed as an alternative to the so-called induced-current method, which is based on an approximation for the electric current distribution on the reflector surface. The fields in the aperture plane can be thought of as an ensemble of Huygens sources. The radiation pattern can be computed via a numerical summation of these sources.

### 3. EQUIVALENCE PRINCIPLE

The ability to determine electromagnetic waves radiated fields via field equivalence principles is a useful concept, and the development can be traced by to Schelkunoff [2]. The equivalence principle often makes an exact solution easier to obtain or suggests approximate methods that are of value in simplifying antenna problems. Field equivalence principles are treated at length in the literature, and we will not consider the many variants here. The book by Collin and Zucker [11] is a useful source of reference. The basic concept is illustrated in Fig. 5, where the electromagnetic source region is enclosed by a surface \( S \) that is referred to as a Huygens surface.

In essence, Huygens’ principle and the equivalence theorem shows how to replace actual sources by a set of equivalent sources spread over the surface \( S \) [12]. The equivalence principle is developed by considering a radiating source, electrically represented by current densities \( J_1 \) and \( M_1 \). Assume that the source radiates fields \( E_1 \) and \( H_1 \) everywhere. We would like to develop a method that will yield the fields outside the closed surface. To accomplish this, a closed surface \( S \) is shown by the dashed lines that enclose the current densities \( J_1 \) and \( M_1 \). The volume inside \( S \) is denoted by \( V \). The primary task is to replace the original problem (Fig. 5a) by an equivalent one that will yield the same fields \( E_1 \) and \( H_1 \) (Fig. 5b). The formulation of the problem can be greatly facilitated if the closed surface is judiciously chosen so that the fields over most of—if not the entire—surface are known a priori.

The original sources \( J_1 \) and \( M_1 \) are removed, and we assume that there exists a field \( E, H \) inside \( V \). For this field to exist within \( V \), it must satisfy the boundary conditions on the tangential electric and magnetic field components on surface \( S \). Thus on the imaginary surface \( S \), there must exist equivalent sources [13]:

\[
J_s = n \times (H_1 - H) \tag{4}
\]

\[
M_s = -n \times (E_1 - E) \tag{5}
\]

These equivalent sources radiate into an unbounded space. The current densities are said to be equivalent

![Figure 4. Electromagnetic field versus distance along the Y axis, with \( R_0 = 2.5(a), 5(b), 15(c), \) and 20(d) cm.](image)

![Figure 5. Equivalence principle with a closed Huygens surface \( S \) enclosing sources: (a) original problem; (b) equivalent problem.](image)
only outside region \( V \), because they produce the original field \( (E_1, H_1) \). A field \( E \) or \( H \), different from the original, may result within \( V \).

The sources for electromagnetic fields are always electrical currents. However, the electrical current distribution is usually unknown. In certain structures, it may be a complicated function, particularly for slots, horns, reflectors, and lenses. With these types of radiators, the theoretical work is seldom based on the primary current distributions. Rather, the results are obtained with the aid of what is known as aperture theory [14]. This theory is based on the fact that an electromagnetic field in a source-free, closed region is completely determined by the values of tangential \( E \) or tangential \( H \) fields on the surface of the closed region. For exterior regions, the boundary condition at infinity may be employed to close the region. This is exemplified by the following case.

Without changing the \( E \) and \( H \) fields external to \( S \), the electromagnetic source region can be replaced by a zero-field region with approximate distributions of electric and magnetic currents \( (J_s \text{ and } M_s) \) on the Huygens surface. This example is overly restrictive and we could specify any field within \( S \) with a suitable adjustment. However, the zero-internal-field approach is particularly useful when the tangential electric fields over a surface enclosing the antenna are known or can be approximated. In this case, the surface currents can be obtained directly from the tangential fields, and the external field can be determined.

Assuming zero internal fields, we can consider the electromagnetic sources inside \( S \) to be removed, and the radiated fields outside \( S \) are then determined from the electric–magnetic surface current distributions alone. This offers significant advantages when the closed surface is defined as a two-hemisphere region, with all sources contained on only one side of the plane. If either the electric or magnetic field arising from these sources can be determined over the planar Huygens surface \( S \), then the radiated fields on the far side of the plane can be calculated. The introduction of an infinite conducting sheet just inside of the Huygens surface here will not complicate the calculations of the radiated fields in the other half-space [15]. This infinite plane model is useful for antennas the radiation of which is directed into the right hemisphere (Fig. 6), and has found wide application in dealing with aperture antennas. For instance, if the antenna is a rectangular horn, it is assumed that the horn transitions into an infinite flange. All tangential fields outside the rectangular boundary along the infinite Huygens surface are taken to be zero.

When the limitations of the half-space model are acceptable, it offers the important advantage that either the electrical or magnetic currents need to be specified. However, knowledge of both is not required. It must be emphasized that any of the methods described before will produce exact results over the Huygens surface. In the analysis of electromagnetic problems, often it is easier to form equivalent problems that will yield the same solution only within a region of interest.

The steps that must be used to form an equivalent problem and solve an aperture antenna problem are as follows:

1. Select an imaginary surface that encloses the actual sources (the aperture). The surface must be judiciously chosen so that the tangential components of the electric field and/or the magnetic field are known, exactly or approximately, over its entire span. Ideally, this surface flat plane extending to infinity.

2. Over the imaginary surface \( S \), form equivalent current densities \( J_s \) and \( M_s \), assuming that the \( E \) and \( H \) fields within \( S \) are not zero.

3. Finally, solve the equivalent-aperture problem.

4. RECTANGULAR APERTURES

There are many kinds of antennas for which the radiated electromagnetic fields emanate from a physical aperture. This general class of antennas provides a very convenient basis for analysis that permits a number of well-established mathematical techniques to be applied and results in expressions for the distant radiation fields.

Horn and parabolic reflectors can be analyzed as aperture antennas. Incident fields are replaced by equivalent electric and magnetic currents. With use of vector potentials, the far fields are found as a superposition of each source. Generally, one can assume that the incident field is a propagating free-space wave, the electric and magnetic fields of which are proportional to each other. This will give the Huygens source approximation and allow us to use integrals of the electric field in the aperture plane. Each point in the aperture is considered as a source of radiation.

The first step involved in the analysis of aperture antennas is to calculate the electromagnetic fields over the aperture due to the sources on the rearward side of the infinite plane and to use these field distributions as the basis for the prediction of the distance fields on the forward half-space. The electromagnetic fields in the aperture plane cannot be determined exactly, but approximation distributions can be found by many different methods, which are dependent on the antenna. One can
find the far-field radiation pattern for various distributions by, for instance, a Fourier transform relation. Consider a line source of length $L_w$ using the coordinate system illustrated in Fig. 7. Assume that the source is positioned in a ground plane of infinite extent. This model is simple and yet the analysis gives results that demonstrate the main features of the one-dimensional aperture. The line-source distribution does show a practical realization, namely, in a long one-dimensional geometry that has sufficient elements to enable it to be approximated by a continuous distribution. The applicable transform is [7,16]

$$E(y) = \int_{-\infty}^{\infty} E(x) e^{ikx \sin \theta} dx$$  \hspace{1cm} (6)$$

and the inverse transform is

$$E(x) = \int_{-\infty}^{\infty} E(\sin \theta) e^{-ikx \sin \theta} d(\sin \theta)$$  \hspace{1cm} (7)$$

where $k = 2\pi/\lambda$. For real values of $\theta$, $-1 < \sin \theta < 1$, the distribution represents radiated power, while outside this region it represents reactive or stored power [17]. The field distribution $E(\sin \theta)$, or an angular spectrum, refers to a distribution of plane waves. The angular spectrum finite aperture is the same as in the far-field pattern, $E(\theta)$. Now for a finite aperture, the Fourier integral representation [Eq. (6)] may be rewritten as [8] follows:

$$E(\theta) = \int_{-L_w/2}^{L_w/2} E(x) e^{ikx \sin \theta} dx$$  \hspace{1cm} (8)$$

For example, consider a so-called uniform distribution in which the amplitude is constant over the aperture region $-L_w/2 \leq x \leq L_w/2$:

$$E(x) = \frac{1}{L_w}$$  \hspace{1cm} (9)$$

$E(x) = 0$ outside this region. The radiation pattern can be found by incorporating this into Eq. (8):

$$E(\theta) = \frac{1}{L_w} \int_{-L_w/2}^{L_w/2} e^{(2\pi x/\lambda) \sin \theta} dx$$  \hspace{1cm} (10)$$

We complete this straightforward integration and get the final result:

$$E(\theta) = \frac{\sin \frac{\pi L_w \sin \theta}{\lambda}}{\pi L_w \sin \theta}$$  \hspace{1cm} (11)$$

This $\sin(\pi x)/x$ distribution is very important in antenna theory and is the basis for many practical antenna designs. It results in the narrowest radiation pattern beamwidth of any other distribution, but on the other hand has the highest first sidelobe level of $-13.2$ dB.

Another popular continuous aperture distribution is the cosine raised to a power $n$ distribution. For all $-L_w/2 \leq x \leq L_w/2$, let

$$E(x) = \cos^n \frac{\pi x}{L_w}$$  \hspace{1cm} (12)$$

and $E(x) = 0$ for all $x$ outside this region. This popular distribution is illustrated in Fig. 8 for $n = 1, 2, 3$. To make a relative comparison of the two types of line-source...
distribution discussed, we must first normalize the distributions to ensure that the total radiated power is the same relative to the uniform case. To do this, we multiply the radiation pattern expressions by the normalization constant:

$$C_p = \frac{1}{\int_{-L_w/2}^{L_w/2} E^2(x)dx}$$  \hspace{1cm} (13)

To demonstrate, we computed the antenna radiation pattern for a one-meter-long line-source distribution for uniform, cosine, and cosine$^2$ distributions. For an assumed wavelength of 3 cm, the resulting antenna radiation patterns are shown in Fig. 9. These data indicate that the more heavily tapered the illumination, the greater the decrease in sidelobe levels, but this occurs for a penalty in reduced mainlobe directivity.

Many distributions actually obtained in practice can be approximated by one of the simpler forms or by a combination of simple forms. For example, a common linear aperture distribution is the cosine on a pedestal $p$:

$$E(x) = p + (1 - p) \cos \frac{\pi x}{L_w}$$  \hspace{1cm} (14)

where $0 \leq p \leq 1$. This is a combination of a uniform plus a cosine-type distribution. The so-called triangular distribution is of interest

$$E(x) = 1 + \frac{x}{L_w/2}$$  \hspace{1cm} (15)

for $-L_w/2 \leq x \leq 0$, and

$$E(x) = 1 - \frac{x}{L_w/2}$$  \hspace{1cm} (16)

for $0 \leq x \leq L_w/2$.

In practice, the rectangular aperture is a common microwave antenna shape. Because of its configuration, the rectangular coordinate system shown in Fig. 10 is the
most convenient system in which to express the electromagnetic fields at the aperture. Here, the aperture lies in the $x$-$y$ plane and has a defined tangential aperture distribution $E(x,y)$. In keeping with the equivalence principle, we shall assume that the $x$-$y$ plane is a closed surface that extends from $-\infty$ to $+\infty$ in the $x$-$y$ plane. Outside the rectangular aperture boundaries we shall assume that the field distribution is zero for all points on the infinite surface. The task is then to find the fields radiated by the aperture; specifically, the pattern mainlobe beamwidths, the first sidelobe levels, and the directivity. Also shown in the figure is the superposition of a standard spherical coordinate system that will allow us to define a radiation pattern in terms of the two angle coordinates $\theta, \phi$.

Assuming a rectangular aperture of dimension $A_w$ in the $x$ plane and $B_w$ in the $y$ plane, the radiation pattern may then be obtained from the integral [18]

$$E(\theta, \phi) = \int_{-B_w/2}^{B_w/2} \int_{-A_w/2}^{A_w/2} E(x,y) e^{i(k_x x + k_y y)} \, dx \, dy$$

(17)

in which the directional wavenumbers are given by

$$k_x = k \sin \theta \cos \phi$$

$$k_y = k \sin \theta \sin \phi$$

These are also known as the $x$ and $y$ components of the propagation vector $k$ [19].

For many types of antennas such as the rectangular horn antenna, the $x$ and $y$ functions are separable and may be expressed in the following form:

$$E(x,y) = E(x)E(y)$$

(18)

For these distributions, the pattern in the principal $x$-$z$ plane can be determined from the line-source distribution $E(x)$ while the pattern in the $y$-$z$ plane can be determined from the line-source distribution $E(y)$. We can illustrate this by assuming that both $E(x)$ and $E(y)$ are uniform distributions in which $E(x) = 1/A_w$ and $E(y) = 1/B_w$. We enter these into the Fourier transform relationship of Eq. (17) to get the following result:

$$E(\theta, \phi) = \frac{\sin k_x A_w}{k_x A_w} \frac{\sin k_y B_w}{k_y B_w}$$

(19)

From this pattern, we conclude that the principal plane radiation patterns of separable rectangular distributions correspond to the respective line-source distributions.

Next, we have applied this to an aperture size of $A_w = 75$ cm in the $x$ dimension and $B_w = 125$ cm in the $y$ dimension, an operating wavelength of 3 cm, and for simple cosine distributions in each plane. The results are plotted in Fig. 11, where it should be pointed out that $\phi = 0^\circ$ corresponds to the principal-plane radiation pattern in the $x$-$z$ plane, $\phi = 90^\circ$ is the principal-plane pattern in the $y$-$z$ plane, and $\phi = 45^\circ$ corresponds to the principal-plane pattern in the intercardinal plane. For nonseparable distributions, the integration of Eq. (17) is best carried out on a PC computer using numerical methods.

To generalize, we have applied a computer code to compute the secondary radiation patterns produced by uniform cosine raised to power $n$, cosine on a pedestal $p$, and triangular rectangular aperture distributions. The results shown in Table 1 compare the gain, mainlobe beamwidth, and the first sidelobe levels for each. All gain levels are compared with the uniform illumination case, and total radiated power is assumed in each case.

A uniform line-source or rectangular aperture distribution produces the highest directivity. However, the first sidelobe is only about $-13.2$ dB down. The results also show that the first sidelobe levels for a cosine illuminated aperture are $-23$ dB down but at a penalty of $-0.91$ dB in reduction of directivity compared to the uniform illuminated case. Other distributions have even lower first sidelobe levels but even greater reduction in directivity compared to the uniform illuminated case. Thus, aperture distributions used in practice must be a tradeoff or compromise between the desired directivity (or gain) and first sidelobe level.

**5. CIRCULAR APERTURES**

Circular aperture shape antennas form the largest single class of aperture antennas. For instance, the circular parabolic reflector is used extensively in satcom (satellite communications), terrestrial telecommunications, and radar applications.

The coordinate system used to analyze the radiation from a circular aperture of diameter $D_w$ is the spherical coordinate system shown in Fig. 12, where the aperture lies in the $x$-$y$ plane and radiation can be described in terms of the spherical coordinate components $\theta$ and $\phi$. The radiation pattern from a circular aperture can be calculated by applying Huygens’ principle in much the same way as we did for rectangular apertures. The simplest form of a circular aperture distribution is one in which the field does not vary with $\phi$, that is, one that is rotationally
symmetric. This is not always true in practice; however, we will assume that to be the case here in order to demonstrate the methodology of analyzing circular apertures.

As was the case with rectangular apertures, a Fourier transform relationship exists between the antenna distribution and the far-field radiation pattern. For a circular symmetric aperture distribution, the radiation pattern can be written in normalized form [6]

\[
E(u) = \frac{1}{n^2} \int_{0}^{\infty} \int_{0}^{\pi} E(p) e^{i p u \cos(\theta - \phi')} p \, dp \, d\phi' \tag{20}
\]

where

\[
u = \frac{D_w \sin \theta}{\lambda}
\]

and the normalized radius is

\[
p = \frac{2 \pi r}{D_w}
\]

For a uniformly illuminated circular aperture, the normalized radiation pattern is expressed in terms of a first-order Bessel function

\[
E(u) = \frac{2 J_1(nu)}{\pi u} \tag{21}
\]

which, in turn, can be expressed as

\[
E(\theta) = \frac{2 J_1(\frac{\pi D_w \sin \theta}{\lambda})}{\frac{\pi D_w \sin \theta}{\lambda}} \tag{22}
\]

The uniformly illuminated circular aperture radiation pattern has a first sidelobe level of \(-17.6 \text{ dB}\) compared with \(-13.2 \text{ dB}\) for the uniformly illuminated rectangular aperture or line source. Other types of circular aperture distributions include the cosine raised to a power \(n\)

\[
E(r) = \cos^n \frac{\pi r}{D_w} \tag{23}
\]

where \(0 \leq r \leq D_w/2\).

**Table 1. Radiation Pattern Characteristics Produced by Various Linear Aperture Distributions**

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Comments</th>
<th>Normalized Half-Power Beamwidth (deg) HPbw/K</th>
<th>Normalized Null-to-Null Beamwidth (deg) NULLbw/K</th>
<th>Sidelobe Level (dB): SLL</th>
<th>Normalized Sidelobe Angle (deg): SLpos</th>
<th>Gain Relative to Uniform (dB): Go</th>
<th>Power Gain Factor Relative to Uniform Go</th>
<th>Voltage Gain Factor Relative to Uniform Go</th>
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<tr>
<td>Uniform</td>
<td></td>
<td>50.67</td>
<td>114.67</td>
<td>-13.26</td>
<td>82.00</td>
<td>0.00</td>
<td>1.000</td>
<td>1.000</td>
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<tr>
<td>Cosine raised to power (n)</td>
<td>(n = 1)</td>
<td>68.67</td>
<td>172.00</td>
<td>-23.00</td>
<td>108.33</td>
<td>-0.91</td>
<td>0.810</td>
<td>0.900</td>
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<tr>
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<td>-2.89</td>
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<td>172.00</td>
<td>-23.01</td>
<td>108.33</td>
<td>-0.91</td>
<td>0.810</td>
<td>0.900</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
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<td>162.00</td>
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<td>(p = 0.2)</td>
<td>62.00</td>
<td>152.67</td>
<td>-21.66</td>
<td>97.00</td>
<td>-0.50</td>
<td>0.892</td>
<td>0.944</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.3)</td>
<td>59.33</td>
<td>144.67</td>
<td>-20.29</td>
<td>93.67</td>
<td>-0.35</td>
<td>0.923</td>
<td>0.961</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.4)</td>
<td>58.00</td>
<td>138.00</td>
<td>-18.92</td>
<td>90.67</td>
<td>-0.24</td>
<td>0.947</td>
<td>0.973</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.5)</td>
<td>56.00</td>
<td>132.67</td>
<td>-17.65</td>
<td>88.33</td>
<td>-0.15</td>
<td>0.966</td>
<td>0.983</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.6)</td>
<td>54.67</td>
<td>127.67</td>
<td>-16.53</td>
<td>86.67</td>
<td>-0.09</td>
<td>0.979</td>
<td>0.989</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.7)</td>
<td>54.00</td>
<td>123.33</td>
<td>-15.55</td>
<td>85.00</td>
<td>-0.05</td>
<td>0.989</td>
<td>0.995</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.8)</td>
<td>52.67</td>
<td>120.00</td>
<td>-14.69</td>
<td>83.83</td>
<td>-0.02</td>
<td>0.995</td>
<td>0.998</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 0.9)</td>
<td>52.00</td>
<td>117.33</td>
<td>-13.93</td>
<td>82.67</td>
<td>-0.00</td>
<td>0.998</td>
<td>0.999</td>
</tr>
<tr>
<td>Cosine on a pedestal (p)</td>
<td>(p = 1.0)</td>
<td>50.67</td>
<td>114.67</td>
<td>-13.26</td>
<td>82.00</td>
<td>0.00</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Triangular</td>
<td></td>
<td>73.34</td>
<td>114.59</td>
<td>-26.52</td>
<td>164.00</td>
<td>-1.25</td>
<td>0.749</td>
<td>0.865</td>
</tr>
</tbody>
</table>

**Figure 12.** Coordinate system used to analyze a circular aperture of diameter \(D_w\).
The so-called cosine on a pedestal \( p \) distribution is defined here as

\[
E(r) = p + (1 - p) \cos \frac{\pi r}{D_w} \tag{24}
\]

and the parabolic raised to a power \( n \) distribution is [15]

\[
E(r) = \left[1 - \left(\frac{r}{D_w/2}\right)^2\right]^n \tag{25}
\]

To analyze the various circular aperture distributions, we can utilize a PC computer using numerical methods to perform the aperture integration of Eq. (20). To demonstrate the behavior of various distributions discussed, a computer code was applied to compute the secondary pattern characteristic produced by a uniform cosine raised to a power \( n \), cosine on a pedestal \( p \), and parabolic raised to power \( n \) distributions. The results shown in Table 2 compare the gain, beamwidth, and the first sidelobe levels of each. All gain levels are compared with the uniform illumination case.

### 6. BEAM EFFICIENCY

This discussion considers the effect of the aperture field distribution on the beam and aperture efficiencies. For many applications, the fraction of the total radiated energy that is in the main (null-to-null) antenna beam is important. This quantity, called the beam efficiency [20], can be used to judge the ability of the antenna to discriminate between signals received through its mainlobe and those through the minor lobes.

Before we delve into this subject, it is helpful to review some fundamentals. The mainbeam consists of the solid angle of the mainlobes in the two principal planes, with minor lobes neglected. The total-beam solid angle \( \Omega_m \) consists of the mainbeam solid angle plus the minor-lobe solid angle. Furthermore, the ratio of the mainbeam solid angle to the total-beam solid angle defines a property called the beam efficiency \( \eta_b \):

\[
\eta_b = \frac{\Omega_m}{\Omega_a} \tag{27}
\]
In terms of the radiated intensity \( E(\theta, \phi) \) of a pencil beam with boresight at \((\theta = 0, \phi = 0)\), the beam efficiency can be defined by [13]

\[
\eta_b = \frac{\int_{0}^{\theta_n/2} \int_{\phi_n/2}^{0} E(\theta, \phi) E(\theta, \phi)^* \sin \theta d\phi d\theta}{\int_{0}^{\pi/2} \int_{0}^{\pi/2} E(\theta, \phi) E(\theta, \phi)^* \sin \theta d\phi d\theta} \tag{28}
\]

where \( \theta_n \) and \( \phi_n \) are the null-to-null beamwidths in the two principal planes. Also, \( E(\theta, \phi)^* \) denotes the conjugate of \( E(\theta, \phi) \).

The directivity of the aperture antenna can be expressed as

\[
D = \frac{4\pi}{\Omega_a} = \frac{4\pi A_p}{\lambda^2} \tag{29}
\]

where \( A_p \) is the physical area of the aperture. The aperture efficiency is defined as the ratio of the effective aperture area \( A_e \) to the physical aperture area, or

\[
\eta_a = \frac{A_e}{A_p} \tag{30}
\]

so that the ratio of the aperture and beam efficiencies is [8]

\[
\frac{\eta_a}{\eta_m} = \frac{A_e \Omega_a}{A_p \Omega_m} \tag{31}
\]

where \( \Omega_a \) is the mainbeam solid angle and \( \Omega_m \) is the total-beam solid angle, both of which are measured in steradians (sr). It is important to recognize, then, that the beam efficiency and aperture efficiency are related to each other.

In general, the aperture and beam efficiencies must be multiplied by a gain degradation factor due to phase errors within the aperture given by [21]

\[
\eta_{pe} = e^{-(2\pi\delta/\lambda)^2} \tag{32}
\]

where \( \delta \) is the RMS phase error over the aperture. It is assumed that the correlation intervals of the deviations are greater than the wavelength. The controlling effect of tapers on the beam and aperture efficiencies tends to decrease them as the phase error increases. The efficiencies are also reduced by the presence of other phase errors.

The curves in Fig. 13 show that the beam efficiency tends to increase with an increase in taper but the aperture efficiency decreases. Maximum aperture efficiency occurs for a uniform aperture distribution, but maximum beam efficiency occurs for a highly tapered distribution. In most cases a taper is used that is intermediate between the two extremes.

### 7. APERTURE SYNTHESIS

To demonstrate the principles in aperture synthesis, consider a one-dimensional line source of length \( L_w \). Earlier, in Eqs. (6) and (7), a Fourier transform pair was defined relating the aperture distribution of a line source with the far-field radiation pattern. In the synthesis process, we wish to determine an aperture distribution that will result in a desired radiation that is not necessarily a symmetric beam; for instance, let us determine the well-known \( \csc^2 \theta \) (cosecant-squared) shaped beam used in a ground-mapping radar. In order to do this, we first express the illumination function as a sum of \( N \) uniform distributions that have relative weights \( c_m \) and a relative linear aperture phase function \( \phi_m \). This may be expressed mathematically as follows:

\[
E(x) = \sum_{m=1}^{N} c_m e^{i\phi_m x} \tag{33}
\]

The Fourier transform of this aperture distribution gives us an expression for the far-field radiation pattern:

\[
E(\theta) = \int_{-L_w/2}^{L_w/2} \sum_{m=1}^{N} c_m e^{i(k \sin \theta + \phi_m) x} dx \tag{34}
\]

which may be expressed in the form

\[
E(\theta) = \sum_{m=1}^{N} c_m \sin \left( \frac{(k \sin \theta + \phi_m) L_w}{2} \right) \left( \frac{k \sin \theta + \phi_m}{2} \right) \tag{35}
\]

Thus, each coefficient \( c_m \) is responsible for a \( \sin x/x \) type of beam, and there are \( N \) different beams. The coefficients may be obtained manually by estimating the number of independent beams and their relative magnitudes and positions in angle space needed to approximate the desired radiation pattern. The results may be extended by the reader to a two-dimensional aperture.

The preceding equations form the basis for Woodward’s aperture synthesis technique [6,28], which quantifies the aperture illumination required to produce a given beam shape that is desired. Another antenna synthesis method
called the virtual array synthesis method was more recently published by Vaskelainen [22]. In this method, the geometry of the virtual array is chosen so that there will be a suitable synthesis method for that geometry, and the synthesis of the virtual array can be done accurately. The excitation values for the virtual array are transformed into the excitation values of the actual array geometry. Matrix operations are simple and large arrays can easily be synthesized. Further information on antenna pattern synthesis techniques is given in Refs. 23–27.

8. MODERN FULL-WAVE METHODS [29]

Some aperture antennas can be addressed with analysis approaches known as full-wave methods. The application of such methods has rapidly expanded with the explosion of high-power PC computers.

Analysis methods are called “full-wave” when they start with the fundamental equations of electromagnetics and discretize them such that they can be reduced to linear matrix equations suitable for solving by a computer. The advantage is that there are no approximations in principle, only the size of the discrete interval, which is usually between 10 and 20 intervals per wavelength. There are three primary full-wave methods used in electromagnetics: the finite-element method (FEM) [30–33], the method of moments (MoM) [34–36], and the finite-difference time-domain (FDTD) method [37,38]. The MoM discretizes Maxwell’s wave equations in their integral form, the finite-difference method (FDM) discretizes the equations in the differential form, and the FEM method discretizes the equations after casting them in a variational form. All three techniques have been applied to aperture antenna analysis [39–41].

The MoM method finds natural application to antennas because it is based on surfaces and currents, whereas the other two methods are based on volumes and fields. This means that for MoM, only the antenna aperture surface structure must be discretized and solved, whereas for FEM and FDTD, all volumes of interest must be discretized. For antenna radiation, the far field would require an inordinate amount of space were it not for the recent development of absorbing boundary conditions. These boundary conditions approximate the radiation conditions of infinite distance in the space very near the radiating structure.

The MoM works by solving for currents on all surfaces in the presence of a source current or field. The radiated field is then obtained by integration of these currents in much the same way as it was obtained in the physical optics (PO) approaches. Thus, the MoM can be applied to any aperture antenna that the PO technique can be applied to, unless the problem is too large for the available computer resources. Ensemble [42] is a commercially available software package that is a 2.5D (two and one-half-dimensional) MoM program used primarily for patch antennas or antennas that can be modeled as layers of dielectrics and conductors. If the top layer is a conductor with radiating holes, the holes are aperture antennas, which this program is designed to analyze.

There is another full-wave commercial software package that is widely used for aperture antenna problems: the high-frequency structure simulator (HFSS) [43]. This is a 3D FEM software package with extensive modeling and automatic meshing capability. It is best for horn antennas or other kinds of antennas formed by apertures in various nonlayered structures. The latest version uses the “perfectly matched layer” type of absorbing boundary conditions.

In practice, full-wave methods cannot be directly applied to high-gain aperture antennas such as reflectors or lenses without difficulties because these structures are usually many wavelengths in size, requiring a large amount of computational resources. Often, however, if there is symmetry in the problem that can be exploited, the number of unknowns for which to solve can be greatly reduced. For instance, a high-gain reflector antenna that has circular symmetry allows for body-of-revolution (BoR) symmetry [44,45] simplifications in the modeling. Similarly, a large lens requires a computer program with dielectric capability [46] in addition to BoR symmetry modeling.

BIBLIOGRAPHY

APPLICATION OF WAVELETS TO ELECTROMAGNETIC PROBLEMS

1. INTRODUCTION

Since the early nineteenth century, Fourier analysis has played an important role in almost all branches of science and engineering and in some areas of social science as well. In this method a function is transformed from one domain to another where many characteristics of the function are revealed. One usually refers to this transform
domain as the spectral, frequency, or wavenumber domain, while the original domain is referred to as time or spatial domain. In many applications, combined time–frequency analysis of a signal provides useful information about the physical phenomena; information that could not be extracted by either the time-domain or the frequency-domain analyses. For instance, in applications to identification and classification of targets based on the analysis of radar echo, time-domain scattering center analysis [1,2] provides information about the local features of the scatterer since these features appear as short time pulses. Frequency-domain analysis of radar echo using the singularity expansion method [3,4] provides information about the global features of the target. The combined time–frequency analysis can provide additional information, such as the dispersive nature of the target [5,6] and the dispersive nature of propagation in a transmission line [7].

Another area of interest to the electromagnetics community concerns solving boundary value problems arising from scattering and propagation of electromagnetic waves. Two of the main properties of wavelets vis-à-vis boundary value problems are their hierarchical nature and the vanishing moments properties. Because of their hierarchical (multiresolution) nature, wavelets at different resolutions (scales) are interrelated, a property that makes them suitable candidates for multigrid-type methods for solving partial-differential equations. On the other hand, the vanishing-moment property, causing wavelets, when integrated against a function of certain order, to render the integral zero, is attractive in sparsifying a dense matrix generated by an integral equation.

In applications to discrete datasets, wavelets may be considered as basis functions generated by dilations and translations of a single function. Analogous to Fourier analysis, there are wavelet series (WS) and integral wavelet transforms (IWTs). In wavelet analysis, WS and IWT are intimately related. The IWT of a finite-energy function on the real line is intimately related to the wavelet function

\( \phi(x) = \sum_{k} p_k \phi(2x - k) \)

with \( \{p_k\} \) belonging to the set of square summable biinfinite sequences. The number 2 in (2) signifies “octave levels.” In fact, this number could be any rational number, but we will discuss only octave levels or scales. From (2) we see that the function \( \phi(x) \) is obtained as a linear combination of a scaled and translated version of itself, and hence the term scaling function.

The subspaces \( V_j \) are generated by \( \phi_j(x) := 2^{j/2} \phi(2^j x - k) \); \( j, k \in \mathbb{Z} \), where \( \mathbb{Z} := \{-\ldots,-1,0,1,\ldots\} \). For each scale \( j \), since \( V_j \subset V_{j+1} \), there exists a complementary subspace \( W_j \) of \( V_j \) in \( V_{j+1} \). This subspace \( W_j \), called “wavelet subspace,” is generated by \( \psi_j(x) := 2^{j/2} \psi(2^j x - k) \), where \( \psi \in L^2 \) is called the “wavelet.” From the discussion above, these results follow easily:

\[
\begin{align*}
V_{j+1} \cup W_{j+1} &= V_j \quad j_2 = j_1 + 1 \\
V_j \cap V_{j+1} &= V_j \quad j_1 > j_2 \\
W_j \cap W_{j+1} &= \{0\} \quad j_1 \neq j_2 \\
V_j \cap W_{j+1} &= \{0\} \quad j_1 \leq j_2 
\end{align*}
\]

The scaling function \( \phi \) exhibits lowpass filter characteristics in the sense that \( \phi(0) = 1 \), where a hat over the function denotes its Fourier transform. On the other hand, the wavelet function \( \psi \) exhibits bandpass filter characteristic in the sense that \( \hat{\psi}(0) = 0 \). Later in the article, we will see some examples of wavelets and scaling functions.

2. WAVELET PRELIMINARIES

In this section we briefly describe the basics of wavelet theory to facilitate subsequent discussion on its application. More details on the topic may be found in the literature [8–16].

2.1. Multiresolution Analysis

As pointed out before, multiresolution analysis (MRA) plays an important role in the application of wavelets to boundary value problems. In order to achieve MRA we must have a finite-energy function (square integrable on the real line) \( \phi(x) \in L^2(\mathbb{R}) \), called a scaling function, that generates a nested sequence of subspaces

\[ \{0\} \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \rightarrow L^2 \] (1)

and satisfies the dilation (refinement) equation, namely

\[ \phi(x) = \sum_{k} p_k \phi(2x - k) \] (2)
2.2. Properties of Wavelets

Some of the important properties that we will discuss in this article are given below:

- **Vanishing moment**—a wavelet is said to have a vanishing moment of order \( m \) if

\[
\int_{\infty}^{\infty} x^p \psi(x) dx = 0; \quad p = 0, \ldots, m - 1
\]

All wavelets must satisfy this condition for \( p = 0 \).

- **Orthonormality**—the wavelets \( \{ \psi_{j,k} \} \) form an orthonormal basis if

\[
\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l} \delta_{k,m}; \quad \text{for all } j, k, l, m \in \mathbb{Z}
\]

where \( \delta_{p,q} \) is the Kronecker delta defined in the usual way as

\[
\delta_{p,q} = \begin{cases} 
1 & p = q \\
0 & \text{otherwise}
\end{cases}
\]

The inner product \( \langle f_1, f_2 \rangle \) of two square integrable functions \( f_1 \) and \( f_2 \) is defined as

\[
\langle f_1, f_2 \rangle := \int_{-\infty}^{\infty} f_1(x) f_2^*(x) dx
\]

with \( f_2^*(x) \), representing the complex conjugation of \( f_2 \).

- **Semiorthogonality**—the wavelets \( \{ \psi_{j,k} \} \) form a semiorthogonal basis if

\[
\langle \psi_{j,k}, \psi_{l,m} \rangle = 0; \quad j \neq l; \quad \text{for all } j, k, l, m \in \mathbb{Z}
\]

2.3. Wavelet Transform Algorithm

Given a function \( f(x) \in L^2 \), the decomposition into various scales begins by mapping the function into a sufficiently high-resolution subspace \( V_M \):

\[
L^2 \ni f(x) \rightarrow f_M = \sum_k g_{M,k} \phi(2^M x - k) \in V_M
\]

Now since

\[
V_M = W_{M-1} + V_{M-1}
\]

\[
= W_{M-1} + W_{M-2} + V_{M-2}
\]

\[
= \sum_{n=1}^{N} W_{M-n} + V_{M-N},
\]

we can write

\[
f_M(x) = \sum_{n=1}^{N} g_{M-n}(x) + f_{M-N}(x)
\]

where \( f_{M-N}(x) \) is the coarsest approximation of \( f_M(x) \) and

\[
f_j(x) = \sum_k \alpha_{j,k} \phi(2^j x - k) \in V_j
\]

\[
g_j(x) = \sum_k \psi_{j,k} \psi(2^j x - k) \in W_j
\]

If the scaling functions and wavelets are orthonormal, it is easy to obtain the coefficients \( \{ \alpha_{j,k} \} \) and \( \{ \psi_{j,k} \} \). However for the semiorthogonal case, we need a dual scaling function \( \tilde{\phi} \) and dual wavelet \( \tilde{\psi} \). Dual wavelets satisfy the “biorthogonality condition”:

\[
\langle \psi_{j,k}, \tilde{\psi}_{l,m} \rangle = \delta_{j,l} \cdot \delta_{k,m}, \quad j, k, l, m \in \mathbb{Z}
\]

For the semiorthogonal case, both \( \psi \) and \( \tilde{\psi} \) belong to the same space \( W_j \) for an appropriate \( j \); likewise \( \phi \) and \( \tilde{\phi} \) belong to \( V_j \). One difficulty with semiorthogonal wavelets is that their duals do not have compact support. We can achieve compact support for both \( \phi \) and \( \psi \) if we forgo the orthogonality requirement that \( V_j \perp W_j \). In such a case we get “biorthogonal wavelets” [17] and two MRAs, \( \{ V_j \} \) and \( \{ W_j \} \). In this article we will discuss application of orthonormal and semiorthogonal wavelets only.

3. INTEGRAL EQUATIONS

Integral equations appear frequently in practice, particularly the first-kind integral equations [18] in inverse problems. These equations can be represented as

\[
L_K f = \int_a^b f(x') K(x, x') dx' = g(x)
\]

where \( f(x) \) is an unknown function, \( K(x, x') \) is the known kernel that might be the system impulse response or Green’s function, and \( g(x) \) is the known response function.

3.1. Electromagnetic Scattering

Consider the problem of electromagnetic scattering by an infinitely long metallic cylinder, as shown in Fig. 1. For such a problem, electric surface current \( J_{sw} \) is related to the incident electric field via an integral equation

\[
j_{SW} \int_C J_{sw}(l') G(l, l') dl' = E_z^s(l)
\]

where

\[
G(l, l') = \frac{1}{4\pi} H^{(2)}_{0}(k_0 |l - l'|)
\]

with the wavenumber \( k_0 = 2\pi/\lambda_0 \). The electric field \( E_z \) is the z component of the incident electric field and \( H^{(2)}_{0} \) is the second-kind Hankel function of order 0, and \( \lambda_0 \) is the wavelength in free space. Here, the contour of integration has been parameterized with respect to the chord length.
The field component $E_z$ can be expressed as

$$E_z(l) = E_0 \exp[jk_0(x(l) \cos \phi_i + y(l) \sin \phi_i)] \quad (17)$$

where $\phi_i$ is the angle of incidence.

Scattering from a thin perfectly conducting strip, as shown in Fig. 2a, gives rise to an equation similar to (15). For this case, we have

$$\int_{-b}^{b} J_{sy}(z')G(z,z')dz' = E_z^i(z) \quad (18)$$

where $G(z,z')$ is as given by (16).

As a final example of the scattering problem, consider scattering from a thin wire as shown in Fig. 2b. Here the current on the wire and the incident field are related to each other as

$$\int_{-l}^{l} J(z')K_w(z,z')dz' = -E'(z) \quad (19)$$

where the kernel $K_w$ is given by

$$K_w(z,z') = \frac{1}{4\pi j \omega_0} \frac{\exp(-jk_0R)}{R^2} \times [(1+jk_0R) \times (2R^2 - 3a^2) + k_0^2 a^2 R^2] \quad (20)$$

$$E'(z) = E_0 \sin \theta \exp(jk_0 z \cos \theta) \quad (21)$$

This kernel is obtained by interchanging integration and differentiation in the integro-differential form of Pocklington’s equation and using the reduced kernel distance $R = \sqrt{a^2 + (z-z')^2}$, where $a$ is the radius of the wire [19].

All the equations described thus far have the form of a first-kind integral equation, namely

$$\int_{-a}^{b} f(x')K(x,x')dx' = g(x) \quad (22)$$

where $f$ is the unknown function and the kernel $K$ and the functions $g$ are known. Here the objective is to reconstruct the function $f$ from a set of known data (possibly measured) $g$. The kernel $K$ may be regarded as the impulse response function of the system.

Although we discuss the solution technique for first-kind integral equations only, the method can be extended to second-kind equations [20,21] and higher-dimensional integral equations [22].

3.2. Transmission-Line Discontinuity

As an example of electromagnetic propagation in a transmission line, consider the problem of characterizing one discontinuity of a coplanar waveguide, shown in Fig. 3. The problem can be formulated by separating the configuration of Fig. 3a into two parts with the help of the equivalence principle [23], according to which the slot regions can be replaced by equivalent surface magnetic
enforcing the boundary condition we can write the magnetic field integral equation by matrix equation to be solved for the unknown coefficients. The magnetic field is given by
\[
\mathbf{H}(r) = \mathbf{H}_1(r) - \mathbf{H}_g(r).
\]
The goal is to transform Eq. (14) into a matrix equation
\[
Z \mathbf{v} = \mathbf{g}
\]
where \(Z\) is a two-dimensional matrix, sometimes referred to as the impedance matrix, \(i\) is the column vector of unknown coefficients to represent \(f\), and \(v\) is another column vector related to \(g\). Computation time depends largely on the way we obtain and solve (27). In the following section we describe conventional and wavelet basis functions that are used to represent the unknown function.

### 4.1. Conventional Basis Functions

The unknown function \(f(x)\) can be written as
\[
f(x) = \sum_n i_n b_n(x)
\]
where \(\{b_n\}\) form a complete set of basis functions. These bases may be "global" (entire-domain), extending the entire length \([a, b]\) or be "local" (subdomain), covering only a small segment of the interval, or a combination of both. Some of the commonly used subdomain basis functions are shown in Fig. 4.

For an exact representation of \(f(x)\), we may need an infinite number of terms in the series presented above. However, in practice, a finite number of terms suffice for a given acceptable error. Substituting the series representation of \(f(x)\) into the original equation (14), we get
\[
\sum_{n=1}^{N} i_n L_K b_n \approx g
\]
For the present discussion we will assume \(N\) to be large enough that this representation is exact. Now by taking the inner product of (29) with a set of weighting functions or testing functions \((t_m; m = 1,\ldots,M)\), we get a set of linear equations
\[
\sum_{n=1}^{N} i_n (t_m L_K b_n) = (t_m g); \quad m = 1,\ldots,M
\]

### 4. MATRIX EQUATION GENERATION

In this section, we will attempt to solve integral equations discussed in the last section. The first step in solving any integral or differential equation is to convert these into a matrix equation to be solved for the unknown coefficients.

Figure 3. (a) Short-circuited coplanar waveguide with uniaxial substrate; (b) equivalent problem.

Figure 4. Typical subdomain basis functions: (a) piecewise constant, (b) piecewise linear, (c) piecewise cosine, and (d) piecewise sine functions.
which can be written in the matrix form as

$$[Z_{mn}] \{i_m\} = \{v_m\} \quad (31)$$

where

$$Z_{mn} = \langle t_m, L_K b_n \rangle; \quad m = 1, \ldots, M; \quad n = 1, \ldots, N$$
$$v_m = \langle t_m, g \rangle; \quad m = 1, \ldots, M$$

The solution of the matrix equation gives the coefficients \(\{i_m\}\) and thereby the solution of the integral equations. Two main choices of the testing functions are (1) \(t_m(x) = \delta(x - x_m)\), where \(x_m\) is a discretization point in the domain; and (2) \(t_m(x) = b_n(x)\). In the former case the method is called point matching, whereas the latter method is known as the Galerkin method. The method so described and those to be discussed in the following sections are generally referred to as “method of moments” (MoM) [24]. We will call MoM with conventional bases as “conventional MoM” and the method with wavelet bases, “wavelet MoM.” Observe that the operator \(L_K\) in the preceding paragraphs could be any linear operator—differential as well as integral.

4.2. Wavelet Bases

Conventional bases (local or global), when applied directly to the integral equations, generally lead to a dense (fully populated) matrix \(Z\). As a result, the inversion and the final solution of such a system of linear equations are very time-consuming. In later sections it will be clear why conventional bases give a dense matrix while wavelet bases produce sparse matrices. Observe that conventional MoM is a single-level approximation of the unknown function in the sense that the domain of the function (e.g., \([a, b]\)), is discretized only once, even if we use nonuniform discretization of the domain. Wavelet MoM as we will discuss, on the other hand, is inherently multilevel in nature.

Beylkin et al. [25] first proposed the use of wavelets in sparsifying an integral equation. Alpert et al. [20] used “waveletlike” basis functions to solve second-kind integral equations. In electrical engineering, wavelets have been used to solve integral equations arising from electromagnetic scattering and transmission-line problems [22,26–40]. In what follows we briefly describe four different ways in which wavelets have been used in solving integral equations.

4.2.1. Use of Fast Wavelet Algorithm. In this method, the impedance matrix \(Z\) is obtained via the conventional method of moments using basis functions such as triangular functions, and then wavelets are used to transform this matrix into a sparse matrix [26,27]. Consider a matrix \(W\) formed by wavelets. This matrix consists of the decomposition and reconstruction sequences and their translates. We have not discussed these sequences here, but readers may find these sequences in any standard book on wavelets [e.g., 8–16].

Transformation of the original MoM impedance matrix into the new wavelet basis is obtained as

$$WZW^T \cdot (W^T)^{-1} i = Wv \quad (32)$$

which can be written as

$$Z_w \cdot i_w = v_w \quad (33)$$

where \(W^T\) represents the transpose of the matrix \(W\). The new set of wavelet-transformed linear equations are

$$Z_w = WZW^T \quad (34)$$
$$i_w = (W^T)^{-1} i \quad (35)$$
$$v_w = Wv \quad (36)$$

The solution vector \(i\) is then given by

$$i = W^T(WZW^T)^{-1} Wv \quad (37)$$

For orthonormal wavelets \(W^T = W^{-1}\) and the transformation (32) is “unitary similar.” It has been shown [26,27] that the impedance matrix \(Z_w\) is sparse, which reduces the inversion time significantly. Discrete wavelet transform (DWT) algorithms can be used to obtain \(Z_w\). Readers may find the details of discrete wavelet transform (octave scale transform) in any standard book on wavelets. In some applications it may be necessary to compute the wavelet transform at nonoctave scales. Readers are referred to the literature [7,41,42] for details on such algorithms.

4.2.2. Direct Application of Wavelets. In another method of applying wavelets to integral equations, wavelets are directly applied; that is, first the unknown function is represented as a superposition of wavelets at several levels (scales) along with the scaling function at the lowest level, prior to using Galerkin’s method described before.

In terms of wavelets and scaling functions we can write the unknown function \(f\) in (14) as

$$f(x) = \sum_{j=J_0}^{J_L} \sum_{k=K_1}^{K_j} w_{j,k} \psi_{j,k}(x) + \sum_{k=K_1}^{K_{j_0}} a_{j_0,k} \phi_{j_0,k}(x) \quad (38)$$

where we have used the multiscale property (10).

It should be pointed out here that the wavelets \(\{\psi_{j,k}\}\) by themselves form a complete set; therefore, the unknown function could be expanded entirely in terms of the wavelets. However, to retain only a finite number of terms in the expansion, the scaling function part of (38) must be included. In other words, \(\{\psi_{j,k}\}\), because of their bandpass
filter characteristics, extract successively lower and lower frequency components of the unknown function with decreasing values of the scale parameter $j$, while $\phi_{j_0,k}$, because of its lowpass filter characteristics, retains the lowest frequency components or the coarsest approximation of the original function.

In Eq. (38), the choice of $j_0$ is restricted by the order of the wavelet, while the choice of $j_u$ is governed by the physics of the problem. In applications involving electromagnetic scattering, as a “rule of thumb” the highest scale, $j_u$, should be chosen such that $1/2^{j_u+1}$ does not exceed $0.1\lambda_0$, where $\lambda_0$ is the operative wavelength.

When (38) is substituted in (14), and the resultant equation is tested with the same set of expansion functions, we get a set of linear equations:

$$
\begin{bmatrix}
[Z_{\phi,\psi}] & [Z_{\phi,\psi}] \\
[Z_{\phi,\psi}] & [Z_{\phi,\psi}]
\end{bmatrix}
\begin{bmatrix}
[a_{j_0,k}] \\
[w_{j,n}]
\end{bmatrix}
= 
\begin{bmatrix}
\langle \psi, \phi_{j_0,k'} \rangle \\
\langle \psi, \phi_{j_0,k'} \rangle
\end{bmatrix}
$$

(39)

where the $\psi$ term of the expansion function and the $\phi$ term of the testing function give rise to the $[Z_{\phi,\phi}]$ portion of the matrix $Z$. A similar interpretation holds for $[Z_{\phi,\psi}]$, $[Z_{\psi,\phi}]$, and $[Z_{\psi,\psi}]$.

By carefully observing the nature of the submatrices, we can explain the “denseness” of the conventional MoM and the “sparseness” of the wavelet MoM. Unlike wavelets, the scaling functions discussed in this article do not possess the vanishing moments properties. Consequently, for two pulse or triangular functions $\phi_1$ and $\phi_2$ (usual bases for the conventional MoM and suitable candidates for the scaling functions), even though $\langle \phi_1, \phi_2 \rangle = 0$ for nonoverlapping support, $\langle \phi_1, L_K \phi_2 \rangle$ is not very small since $L_K \phi_2$ is not small. On the other hand, as is clear from the vanishing-moment property (4) of a wavelet of order $m$, the integral vanishes if the function against which the wavelet is being integrated behaves as a polynomial of a certain order “locally.” Away from the singular points the kernel has a polynomial behavior locally. Consequently, integrals such as $\langle L_K \phi_{j,n} \rangle$ and the inner products involving wavelets are very small for nonoverlapping support.

Because of its “total positivity” property [11, pp. 207–209], the scaling function has a “smoothing” or “variation diminishing” effect on a function against which it is integrated. The smoothing effect can be understood as follows. If we convolve two pulse functions, both of which are discontinuous but totally positive, the resultant function is a linear $B$-spline (triangular function) that is continuous. Likewise, if we convolve two linear $B$-splines, we get a cubic $B$-spline that is twice continuously differentiable. Analogous to these, the function $L_K \phi_{j_0,k}$ is smoother than the kernel $K$ itself. Furthermore, because of the MRA properties that give

$$
\langle \phi_{j,k}, \psi_{j,j'} \rangle = 0, \quad j \leq j'
$$

(40)

the integrals $\langle \phi_{j_0,k'}, (L_K \psi_{j,n}) \rangle$ and $\langle \psi_{j_1,n}, (L_K \phi_{j_0,k}) \rangle$ are quite small.

The $[Z_{\phi,\phi}]$ portion of the matrix, although diagonally dominant, usually does not have entries that are very small compared to the diagonal entries. In conventional MoM case, all the elements of the matrix are of the form $\langle \phi_{j,k}, (L_K \phi_{j,k}) \rangle$. Consequently, we cannot, threshold such a matrix in order to sparsify it. In wavelet MoM case, the entries of $[Z_{\phi,\phi}]$ occupy a very small portion ($5 \times 5$ for linear and $11 \times 11$ for cubic spline cases) of the matrix, while the rest contain entries whose magnitudes are very small compared to the largest, entry; hence a significant number of entries can be set to zero without affecting the solution appreciably.

### 4.2.3. Wavelets in Spectral Domain

In the previous section, we have used wavelets in the space domain. The local support and vanishing-moment properties of wavelet bases were used to obtain a sparse matrix representation of an integral equation. In some applications, particularly in spectral-domain methods in electromagnetics, wavelets in the spectral domain may be quite useful. Whenever we have a problem in which the unknown function is expanded in terms of the basis function in the space (time) domain while the numerical computation takes place in the spectral (frequency) domain, we should look at the space–spectral window product in order to determine the efficiency of using a particular basis function. According to the “uncertainty principle,” the space–spectral window product of a square integrable function cannot be less than 0.5; the lowest value is possible only for functions of Gaussian class. Because of the nearly optimal space–spectral window product of the cubic spline and the corresponding semiorthogonal wavelet, the improper integrals appearing in many spectral-domain formulations of integral equations can be evaluated efficiently. This is due to the fact that higher-order wavelets generally have faster decay in the spectral domain. The spectral-domain wavelets have been used to solve the transmission-line discontinuity problem [22].

#### 4.2.4. Wavelet Packets

The discrete wavelet packet (DWP) similarity transformation has been used to obtain a higher degree of sparsification of the matrix than is achievable using the standard wavelets [38]. It has also been shown that the DWP method gives faster matrix-vector multiplication than do some of the fast multipole methods.

In the standard wavelet decomposition process, first we map the given function to a sufficiently high-resolution subspace ($V_M$) and obtain the approximation coefficients $\{a_{M,k}\}$ (see Section 2). The approximation coefficients $\{a_{M−1,k}\}$ and wavelet coefficients $\{w_{M−1,k}\}$ are computed from $\{a_{M,k}\}$. This process continues; that is, the coefficients for the next-lower level $M−2$ are obtained from $\{a_{M−1,k}\}$, and so on. Observe that in this scheme, only approximation coefficients $\{a_{j,k}\}$ are processed at any scale $j$; the wavelet coefficients are merely the outputs and remain untouched. In a wavelet packet, the wavelet coefficients are also processed, which, heuristically, should result in a higher degree of sparsity since in this scheme, the frequency bands are further divided compared with the standard decomposition scheme.
4.3. Intervallic Wavelets

Wavelets on the real line have been used to solve integral equations arising from electromagnetic scattering and waveguiding problems. The difficulty with using wavelets on the entire real line is that the boundary conditions need to be enforced explicitly. Some of the scaling functions and wavelets must be placed outside the domain of integration. Furthermore, because of truncation at the boundary, the vanishing-moment property is not satisfied near the boundary. Also, in signal processing uses of these wavelets lead to undesirable jumps near the boundaries. We can avoid this difficulty by periodizing the scaling function as

\[ \phi^p_{j,k} := \sum_l \phi_{j,k}(x + l) \]  

where the superscript “\( p \)” implies periodic case. Periodic wavelets are obtained in a similar way. It is easy to show that if \( \phi(2\pi k) = \delta_{k,0} \), which is generally true for the scaling functions, then \( \sum_k \phi(x - k) = 1 \). If we apply the last relation, which is also known as the “partition of unity” to (41), we can show that \( \{(\phi^p_{0,0}) \cup (\psi^p_{j,k}) : j \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, k = 0, \ldots, 2^j - 1 \} \) generates \( L^2([0, 1]) \).

Periodic wavelets have been used in other studies [35–37]. However, as mentioned elsewhere [10, Sect. 10.7], unless the function that is being approximated by the periodized scaling functions and wavelets has the same values at the boundaries, we still have “edge” problems at the boundaries. To circumvent these difficulties, wavelets, constructed especially for a bounded interval, have been introduced in [40]. Details on intervallic wavelets may be found in the literature [40,43–45]. Most of the time, we are interested in knowing the formulas for these wavelets rather than delving into the mathematical rigor of their construction. These formulas may be found in the literature [16,40].

Wavelets on a bounded interval satisfy all the properties of regular wavelets that are defined on entire real line; the only difference is that in the former case, there are a few special wavelets near the boundaries. Wavelets and scaling functions whose support lies completely inside the interval have properties that are exactly same as those of regular wavelets. As an example consider semiorthogonal wavelets of order \( m \). For this case the scaling functions (B-splines of order \( m \)) have support \([0, m]\), whereas the corresponding wavelet extends the interval \([0, 2m - 1]\). If we normalize the domain of the unknown function from \([a, b]\) to \([0, 1]\) then there will be \( 2^j \) segments at any scale \( j \) (discretization step = \( 2^{-j} \)). Consequently, in order to have at least one complete inner wavelet, the following condition must be satisfied:

\[ 2^j \geq 2m - 1 \]  

and \( j = 3 \) are shown in Fig. 6a, while Fig. 6b gives only the corresponding boundary wavelets near \( x = 0 \) and one inner wavelet. The rest of the inner wavelets can be obtained by simply translating the first one whereas the boundary wavelets near \( x = 1 \) are the mirror images of ones near \( x = 0 \).

5. NUMERICAL RESULTS

In this section we present some numerical examples for applications of wavelets to time–frequency analysis and integral equations. The purpose of these results is to give readers an idea of how wavelets can be applied. These examples can certainly be solved by other methods more efficiently, but they help understand wavelet approach in a simpler way.

Because of page limitations, we present only one example from electromagnetic scattering problems, the one for a cylindrical geometry. Numerical results for strip and wire problems can be found in Ref. 31. Results for spectral-domain applications of wavelets to transmission-line discontinuity problems may be found in Ref. 22. For more applications of wavelets to electromagnetic problems, readers may refer to Ref. 39.

The matrix equation (39) is solved for a circular cylindrical surface [40]. The surface current distribution is computed using linear and cubic spline wavelets. The wavelet MoM results are compared with the conventional MoM results. To obtain the conventional MoM results, we have used triangular functions for both expanding the unknown current distribution and testing the resultant equation. The conventional MoM results have been verified with a series solution [23]. The results of the conventional MoM and the wavelet MoM agree very well.

We want to show how “thresholding” affects the final solution. By “thresholding,” we mean setting those elements of the matrix to zero that are smaller (in magnitude) than some positive number \( \delta (0 \leq \delta < 1) \), called the threshold parameter, times the largest element of the matrix.

Let \( z_{\text{max}} \) and \( z_{\text{min}} \) be the largest and the smallest elements of the matrix in (39). For a fixed value of the threshold parameter \( \delta \), define percent relative error (\( \varepsilon_{\delta} \)) as

\[ \varepsilon_{\delta} := \frac{|f_0 - f_\delta|_2}{|f_0|_2} \times 100 \]  

and percent sparsity (\( S_{\delta} \)) as

\[ S_{\delta} := \frac{N_0 - N_{\delta}}{N_0} \times 100 \]  

where \( f_\delta \) represents the solution obtained from (39) when the elements whose magnitudes are smaller than \( \delta z_{\text{max}} \) have been set to zero. Similarly, \( N_{\delta} \) is the total number of elements left after thresholding. Clearly, \( f_\delta(x) = f(x) \) and \( N_0 = N^2 \), where \( N \) is the number of unknowns.
Table 1 gives an idea of the relative magnitudes of the largest and the smallest elements in the matrix for conventional and wavelet MoM. As is expected, because of their higher vanishing moment property, cubic spline wavelets give the higher ratio, $z_{\text{max}}/z_{\text{min}}$.

The matrix elements with $\delta = 0.0002$ for the linear spline case are shown in Fig. 7. Figures 8 and 9 give an idea of the pointwise error in the solution for linear and cubic spline cases for different values of threshold parameter.

It is worth pointing out here that regardless of the size of the matrix, only $5/5$ in the case of the linear spline and $11 \times 11$ in the case of the cubic splines remain unaffected by thresholding; a significant number of the remaining elements can be set to zero without causing much error in the solution.

6. SEMIORTHOGONAL VERSUS ORTHOGONAL WAVELETS

Both semiorthogonal and orthogonal wavelets have been used for solving integral equations. A comparative study of their advantages and disadvantages has been reported [31]. The orthonormal wavelet transformation, because of its unitary similar property, preserves the condition number $(\kappa)$ of the original impedance matrix $Z$; semiorthogonal wavelets do not. Consequently, the transformed matrix equation may require more iterations to converge to the desired solution. Some preliminary results comparing the condition number of matrices for different cases are given in Table 2 (where ON = orthonormal and SO = semiorthogonal).

In applying wavelets directly to solve integral equations, one of the most attractive features of semiorthogonal wavelets is that closed-form expressions are available [16,40]. Few of the continuous ON wavelets can be written in closed form.

One thing to be kept in mind is that, unlike signal processing applications where one usually deals with a discretized signal and decomposition and reconstruction sequences, here in the boundary value problem we often have to compute the wavelet and scaling function values at any given point. For a strip/thin-wire case, a comparison of the computation time and sparsity is summarized in Tables 3 and 4 [31].

Semiorthogonal wavelets are symmetric and hence have generalized linear phase [11, pp. 160–174], an important factor in function reconstruction. It is well known [10, Sect. 8.1] that symmetric or antisymmetric, real-valued, continuous, and compactly supported ON scaling functions and wavelets do not exist. Finally, in using...
wavelets to solve spectral domain problems, as discussed before, we need to look at the time–frequency window product of the basis. Semiorthogonal wavelets approach the optimal value of the time–frequency product, which is 0.5, very fast. For instance, this value for the cubic spline wavelet is 0.505. It has been shown [46] that this product approaches $N$ with the increase in smoothness of ON wavelets.

7. DIFFERENTIAL EQUATIONS

An ordinary differential equation (ODE) can be represented as

$$L f(x) = g(x); \quad x \in [0, 1]$$  \hspace{1cm} (45)

with

$$L = \sum_{j=0}^{m} a_j(x) \frac{d^j}{dx^j}$$  \hspace{1cm} (46)

and some appropriate boundary conditions. If the coefficients $a_j$ are independent of $x$, then the solution can be obtained via a Fourier method. However, in the ODE case, with nonconstant coefficients, and in PDEs, we generally use finite-element or finite-difference-type methods.

7.1. Multigrid Method

In the traditional finite-element method (FEM), local bases are used to represent the unknown function and the solution is obtained by Galerkin’s method, similar to the approach described in previous sections. For the differential operator, we get sparse and banded stiffness matrices that are generally solved using iterative techniques, such as the Jacobi method.

One disadvantage of conventional FEM is that the condition number ($\kappa$) of the stiffness matrix grows as $O(h^{-2})$, where $h$ is the discretization step. As a result, the convergence of the iterative technique becomes slow and the solution becomes sensitive to small perturbations in the matrix elements. If we study how the error
decreases with iteration in iterative techniques, such as the Jacobi method, we find that the error decreases rapidly for the first few iterations. After that, the rate at which the error decreases slows down [47, pp. 18–21]. Such methods are also called “high-frequency methods” since these iterative procedures have a “smoothing” effect on the high-frequency portion of the error. Once this portion is eliminated, convergence becomes quite slow. After the first few iterations, if we could rediscretize the domain with coarser grids and thereby go to lower frequency, the convergence rate would be accelerated. This leads us to a multigrid-type method.

Multigrid or hierarchical methods have been proposed to overcome the difficulties associated with the conventional method [47–63]. In this technique one performs a few iterations of the smoothing method (Jacobi type), and then the intermediate solution and the operator are projected to a coarse grid. The problem is then solved at the coarse grid, and by interpolation one goes back to the finer grids. By going back and forth between finer and coarser grids, the convergence can be accelerated. It has been shown for elliptic PDEs, that for wavelet-based multilevel methods, the condition number is independent of discretization step [i.e., $\kappa = O(1)$] [58]. The multigrid method is too involved to discuss in this article. Readers are encouraged to look at the references provided at the end of this article.

Multiresolution aspects of wavelets have also been applied in evolution equations [62,63]. In evolution problems, the space and time discretizations are interrelated to gain a stable numerical scheme. The timestep must be determined from the smallest space discretization. This makes the computation quite complex. A spacetime adaptive method has been introduced [63] where wavelets have been used to adjust the spacetime discretization steps locally.

7.2. Multiresolution Time-Domain (MRTD) Method

The explosive growth in wireless communications (3G Cellular Systems, 802.11 WLANs) has spawned a great deal of research in electronic packaging for high-performance devices. Silicon-embedded components, ultracompact efficient antenna technology, and micromachining technology are critical to meet the cost and performance requirements for a higher level of multifunction integration in the development of wireless transceivers, since they can
The multiresolution time-domain (MRTD) technique [62,65] is an adaptive generalization of the FDTD technique that is based on the principles of multiresolution analysis and makes use of wavelets to alleviate the computational burdens of FDTD for complex or large structures, such as multilayer packages or MEMS, where the position of the boundaries is time changing and the membrane thickness is much smaller than any other detail in the transverse direction. The MRTD technique allows the cell resolution to vary with both time and position. The wavelets can be used to represent higher levels of detail along with higher-frequency content. As fields propagate through the structure, the resolution can be varied to allow for the rapidly changing fields.

The multiresolution time-domain (MRTD) technique uses a wavelet discretization of Maxwell’s equations to provide a time- and space-adaptive electromagnetic modeling scheme. The advantage of this method is that it can use much larger cells than similar methods [64], such as finite-difference time-domain (FDTD). The number of basis functions used in each cell can be varied as a function of space and time [71]. In this way, grids of complex structures can use high-resolution cells in areas of large field variation and lower-resolution cells elsewhere. The multiresolution time-domain technique draws its name from the application of multiresolution principles to Maxwell’s equations. In the application of the method, the electric and magnetic fields are expanded into a basis functions used in each cell can be varied as a function of space and time [71].

The multiresolution time-domain technique (MRTD) considerably reduce the MMIC real estate and the amount of needed discrete elements. In addition, advances in device processing are enabling the creation of increasingly compact microwave circuits. These circuits incorporate a high degree of functionality through the combination of many microwave components in close proximity. These advanced devices often utilize geometries with high aspect ratios, small feature size, and moving parts. These characteristics, which are necessary to the operation of these devices, often lead to difficulties in predicting performance. The simulation of these complex devices requires the use of extremely small elements or cells, which can tax many simulation tools beyond their limits. This has led to the use of a combination of methods, such as full-wave simulation and microwave circuit simulation, or, if higher accuracy is required, the use of a parallel full-wave simulator on specialized hardware. For the modeling of all of these wireless elements, time-domain full-wave techniques demonstrate numerous advantages since they are robust and easy to program, and they can use wideband excitations that allow for one simulation to cover the entire frequency band of interest and can be easily parallelized on relatively inexpensive hardware, making it possible to simulate large structures.

The FDTD method is one of the most mature and versatile time-domain numerical techniques and has been used for a wide variety of structures. The use of variable gridding along with effective parallelization approaches allows fine details of large structures to be modeled. Curves and diagonal elements can be modeled using stair stepping. In addition, a wide variety of FDTD enhancements make possible the modeling of small gaps, multidielectric/membrane configurations and resonating passives. Macroscopic results, such as S parameters and impedances, can be determined by probing and comparing voltages and currents at different points in the structure. The multiresolution time-domain technique (MRTD)

<table>
<thead>
<tr>
<th>Basis and Transform</th>
<th>Number of Unknowns</th>
<th>Octave Level</th>
<th>δ</th>
<th>$S_\delta$</th>
<th>$\varepsilon_\delta$</th>
<th>Before Threshold</th>
<th>After Threshold</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pulse and none</td>
<td>64</td>
<td>NA</td>
<td>NA</td>
<td>0.0</td>
<td>2.6 x 10^{-5}</td>
<td>14.7</td>
<td>—</td>
</tr>
<tr>
<td>Pulse and SO</td>
<td>64</td>
<td>1</td>
<td>7.2 x 10^{-2}</td>
<td>46.8</td>
<td>0.70</td>
<td>16.7</td>
<td>16.4</td>
</tr>
<tr>
<td>Pulse and ON</td>
<td>64</td>
<td>1</td>
<td>7.5 x 10^{-3}</td>
<td>59.7</td>
<td>0.87</td>
<td>14.7</td>
<td>14.5</td>
</tr>
</tbody>
</table>

The original impedance matrix is generated using pulse basis functions.

Data not available.

Table 2. Effect of Wavelet Transform Using Semiorthogonal and Orthonormal Wavelets on Condition Number of Impedance Matrix

<table>
<thead>
<tr>
<th>Basis Function</th>
<th>Condition Number $\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spline</td>
<td>0.12</td>
</tr>
<tr>
<td>Semiorthogonal</td>
<td>0.25 x 10^{-3}</td>
</tr>
<tr>
<td>Orthonormal</td>
<td>0.19</td>
</tr>
<tr>
<td>Wire</td>
<td>4.79</td>
</tr>
<tr>
<td>Plate</td>
<td>4.19</td>
</tr>
</tbody>
</table>

Table 3. Comparison of CPU Time (in seconds) per Matrix Element for Spline, Semiorthogonal, and Orthonormal Basis Function [31]

Source: Ref. 31.
stencils decrease the numerical phase error, making it significantly lower than FDTD for low and medium discretizations. Stencil sizes greater than 10 offer a smaller phase error than does FDTD even for discretizations close to 50 cells per wavelength. The enhancement of wavelets further improves the dispersion performance for discretizations close to the Nyquist limit (23 cells per wavelength) making it comparable to that of much denser grids (10–15 cells per wavelength), although it decreases the value of the maximum timestep guaranteeing the stability of the scheme. The finite-domain Haar basis functions (Fig. 10) provide a convenient tool for the transition from FDTD to MRTD due to their compact support, and to their similarity with the FDTD pulse basis, thus providing an effective demonstration tool for this section. In order to create an efficient scheme, Wavelet systems are usually chosen to create sparse discretizations of the modeled equations. The multiresolution time-domain (MRTD) [62] technique makes possible the application of wavelet decomposition principles to the space discretization of Maxwell’s equations. The practical outcome of this application is a timespace-adaptive grid. This grid is very useful because the resolution can be customized to match a given structure using a minimum number of grid points. Through careful application of thresholding [64], this effect can be enhanced by allowing the resolution to be changed based on the requirements of representing the waveform as a function of time. The use of wavelets, however, is not without a price and the application of localized effects becomes difficult.

The quantities that are found directly in MRTD are the values of the wavelet/scaling coefficients; the field values must be reconstructed by summing all coefficients that overlap at any given point. Because all wavelet coefficients cover multiple grid locations, the application of effects at individual points in the grid is challenging. Changing the values of wavelet coefficients to alter the field values at one point effects the values at many other points. Careless coefficient modification can lead to nonphysical field values and unstable algorithms.

The Haar wavelet family is in many ways one of the simplest; however, it has many properties that make its application to practical structures favorable [64]. Most importantly, it is finite-domain and when reconstructed leads to finite areas of constant field value (equivalent gridpoints [66]). Using this property, it is possible to apply pointwise effects in the MRTD grid when an arbitrary level of Haar wavelets is used.

This section discusses the fundamentals of the MRTD derivation and techniques that can be used for the intracell modeling of PECs and dielectric interfaces with Haar MRTD to apply PEC effects at individual equivalent gridpoints. These techniques use wavelet reconstruction/decomposition to apply pointwise effects in the MRTD grid and makes possible the use of the MRTD timespace-adaptive grid for complex structures. Using this method, large, sparse cells can be used in homogeneous areas surrounding high-resolution structures while high-resolution grids can be used to represent fine features. Various examples from large-scale MEMS, packaging, antenna and interconnecting structures demonstrate the potentials of MRTD technique for system-level analysis, design, and optimization.

7.2.1. Fundamentals of MRTD. To clearly present the MRTD method, a brief derivation of 2D Haar MRTD is presented, as well as a partial listing of the properties of

<table>
<thead>
<tr>
<th>Scatterer/Octave Levels</th>
<th>Number of Unknowns</th>
<th>Threshold δ</th>
<th>Sparsity Sd</th>
<th>Relative Error εd</th>
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<tbody>
<tr>
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<td>1 x 10⁻⁶</td>
<td>34.5</td>
<td>4.3 x 10⁻³</td>
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<tr>
<td></td>
<td></td>
<td>5 x 10⁻⁶</td>
<td>34.3</td>
<td>1.3 x 10⁻³</td>
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<td></td>
<td></td>
<td>1 x 10⁻⁵</td>
<td>51.1</td>
<td>16.5</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 x 10⁻⁴</td>
<td>51.6</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5 x 10⁻⁴</td>
<td>69.7</td>
<td>4.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 x 10⁻³</td>
<td>82.4</td>
<td>5.8</td>
</tr>
<tr>
<td>Plate/j = 2, 3, 4</td>
<td>33</td>
<td>1 x 10⁻⁶</td>
<td>28.1</td>
<td>0.7</td>
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<tr>
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<td></td>
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<td>4.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 x 10⁻⁴</td>
<td>69.7</td>
<td>5.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 x 10⁻³</td>
<td>82.4</td>
<td>10.0</td>
</tr>
</tbody>
</table>

Source: Ref. 31.
coefficients for wavelet resolution levels 1 and 2. We assume the scaling function, is 0 \[65\]. Figure 11 presents the wavelet wavelet coefficient, at any resolution level, or with the inner product of each wavelet function with itself is one. The inner product of any wavelet coefficient with any other wavelet coefficient, at any resolution level, or with the scaling function, is 0 \[65\]. Figure 11 presents the wavelet coefficients for wavelet resolution levels 1 and 2. We assume that the maximum used wavelet resolution is \( r_{\text{max}} \).

The Haar basis functions are based on pulses in space. The Haar scaling function \( \phi \), as well as the Haar mother wavelet \( \psi_0 \), are presented in Fig. 10. The scaling function is simply a pulse function over a given domain. The wavelet function is based on the scaling function, and consists of two pulses, each of half the domain of the scaling function and of the opposite magnitude. The inner product of either function with itself is one. The Haar wavelets of higher resolution levels are based on the mother wavelet. For each level of resolution the number of wavelets is doubled while the domain of each is halved. The magnitude of each function is modified so that the inner product of each wavelet function with itself is one. The inner product of any wavelet coefficient with any other wavelet coefficient, at any resolution level, or with the scaling function, is 0 \[65\].

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The reconstruction of the wavelets yields some interesting properties. When the coefficients of the expansion are summed to determine field values, the function appears as a pulsetrain. The pulses have the domain of half of the highest resolution wavelet. Furthermore, these pulses overlap the constant valued sections of the highest-resolution wavelets. A linear combination of the wavelet and scaling functions has as many degrees of freedom as the number of coefficients used. There are \( 2^{r_{\text{max}}} + 1 \) functions used per level, and any finite real value can be represented at the center of each half of the \( r_{\text{max}} \) level wavelets.

The effect of the variable grid when it is used to represent electromagnetic fields can be easily seen. If the field value can be approximated as constant across the half-domain of the highest-resolution wavelet, there is no need for increasing resolution. If the field has more rapid variation, each increase in resolution doubles the effective resolution of the cell. High-resolution cells can be used to represent rapid field variation (such as impressed currents and discontinuity effects) while low-resolution cells can be used elsewhere.

### 7.2.1.1. Haar Basis Functions

#### Haar Basis Functions

Haar basis functions are based on pulses in space. The Haar scaling function \( \phi \), as well as the Haar mother wavelet \( \psi_0 \), are presented in Fig. 10. The scaling function is simply a pulse function over a given domain. The wavelet function is based on the scaling function, and consists of two pulses, each of half the domain of the scaling function and of the opposite magnitude. The inner product of either function with itself is one. The Haar wavelets of higher resolution levels are based on the mother wavelet. For each level of resolution the number of wavelets is doubled while the domain of each is halved. The magnitude of each function is modified so that the inner product of each wavelet function with itself is one. The inner product of any wavelet coefficient with any other wavelet coefficient, at any resolution level, or with the scaling function, is 0 \[65\]. Figure 11 presents the wavelet coefficients for wavelet resolution levels 1 and 2. We assume that the maximum used wavelet resolution is \( r_{\text{max}} \).

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### 7.2.1.2. Haar MRTD Derivation

The equations

\[
\frac{dE_x}{dt} = \frac{1}{\varepsilon} \frac{dH_z}{dy}
\]

(47)

\[
\frac{dE_y}{dt} = -\frac{1}{\varepsilon} \frac{dH_z}{dx}
\]

(48)

\[
\frac{dH_z}{dt} = \frac{1}{\mu} \left[ \frac{dE_x}{dy} - \frac{dE_y}{dx} \right]
\]

(49)

represent the 2D TE\( _z \) mode of Maxwell’s equations for source-free, lossless, isotropic media. These functions will be used to demonstrate the expansion of Maxwell’s equations in this article as a compromise between completeness and space requirements. The expansion of the \( E_x \) field in (47) in terms of Haar scaling and wavelet functions is

\[
E_x(x, y) = \sum_{n, i, j} h_{n(t)} \left[ p E_{i,j}^{x,\phi} \phi_i(x) \phi_j(y) + \sum_{r=0}^{r_{\text{max}}} \sum_{p=0}^{2r-1} p E_{i,j,r,p}^{x,\psi} \psi_{i,p}^r(x) \phi_j(y) + \sum_{r=0}^{r_{\text{max}}} \sum_{p=0}^{2r-1} q E_{i,j,r,p}^{x,\psi} \phi_i(x) \psi_{j,p}^r(y) + \sum_{r=0}^{r_{\text{max}}} \sum_{p=0}^{2r-1} \sum_{q=0}^{2r-1} p E_{i,j,r,p}^{x,\psi} \psi_{i,p}^r(x) \psi_{j,q}^r(y) \right]
\]

(50)

where \( p E_{i,j}^{x,\phi} \) is the coefficient corresponding to scaling function in \( x \) and \( y \) that represents the electric field in the \( i, j \) cell at timestep \( n \). \( \psi_{j,p}^r \) denotes a wavelet of resolution \( r \) at the \( p \) position \( (p - 0.5)/2^r \) of the \( j \) cell. Other coefficients have similar definition. The time dependence is assumed to be constant for each timestep using the pulse \( h_{n(t)} \), although efforts have been published where wavelets have been used in the time domain as well [77].

In a 2D expansion wavelets and scaling functions are used in both the \( x \) and \( y \) directions. The terms in (50) represent the products of the basis functions in both
These equations are written in a matrix form similar to that used in Ref. 65, where, for example, each \( \vec{E}^x_{i,j} \) is the vector of the scaling and wavelet coefficients that represent the electric field in the \( i,j \) cell at timestep \( n \). The \( U \) matrices are the results of the inner products from the method of moments. Equations (51)–(53) form an explicit set of equations that can be used in a time marching scheme similar to the FDTD method [67]. The resolution can be varied on a cell-by-cell basis, and can also be changed as a function of time [64]. The timestep for this method

\[
\Delta t = \frac{1}{c \sqrt{\left( \frac{2r_{\text{max}} + 1}{\Delta x} \right)^2 + \left( \frac{2r_{\text{max}} + 1}{\Delta y} \right)^2}}
\]  

is the same as FDTD for a cell spacing equal to the equivalent gridpoint spacing [64,65].

7.2.2. Subcell Modeling in MRTD. The method presented in the previous section allows a timespace-variable grid to be used to model Maxwell equations. While not presented, it is possible to continuously vary the dielectric constant continuously through a cell [68,79]. Using this method, arbitrary structures consisting of only dielectrics can be modeled efficiently. However, the addition of PEC structures adds difficulties. A novel technique has been introduced that allows for the intracell modeling of multiple PECs [81] using MRTD grids, a development that would eliminate one of the major roadblocks in the minimization of memory requirements using this type of adaptive simulator.

The PEC boundary condition requires that electric fields tangential to PECs be set to zero. In (51)–(53) update equations are presented that allow the determination of wavelet/scaling coefficients at a future timestep based on the wavelet/scaling coefficients of the surrounding fields at previous timesteps. If the PEC structure is the size of an MRTD cell, all the scaling/wavelet coefficients can be zeroed to apply the boundary condition. If the PEC structure is smaller than the cell, however, the scaling/wavelet coefficients must be modified such that the field values at non-PEC locations are unchanged while the field values at PEC locations are zeroed.

One way to determine the scaling/wavelet coefficients that zero selected fields while leaving other fields unchanged is to use the reconstruction matrix. For example, the \( \vec{E}^x_{i,j} \) matrices in (51)–(53) can be transformed into field values by multiplying with a matrix that represents the summation of the fields at the appropriate equivalent gridpoints. In this case

\[
\vec{E}_R = R \vec{E}_W
\]  

where \( \vec{E}_R \) is the reconstructed fields, \( \vec{E}_W \) is the wavelet coefficients, and \( R \) is the reconstruction matrix. It was previously noted that there are as many independent points that can be reconstructed in \( \vec{E}_R \) as there are coefficients in \( \vec{E}_W \). Thus, \( R \) is square. For the case of
The new update equation with PEC locations zeroed is

$$\mathbf{R}_n \mathbf{E}_{i,j}^e = \mathbf{R}_{n-1} \mathbf{E}_{i,j}^e + \frac{\Delta t}{\varepsilon \Delta y} (\mathbf{I}_p^T \mathbf{U}_{E,i}^r \mathbf{R}_{n-1} \mathbf{H}_{i,j}^f)$$

$$+ I_p^T \mathbf{U}_{E,i}^r \mathbf{R}_{n-1} \mathbf{H}_{i,j-1}^f$$

(58)

By multiplying (58) with \( \mathbf{R}^{-1} \) and defining \( \mathbf{U}^p = \mathbf{R}^{-1} \mathbf{I}_p \mathbf{U} \), the PEC MRTD update equation is

$$n\mathbf{E}_{i,j}^e = n-1\mathbf{E}_{i,j}^e + \frac{\Delta t}{\varepsilon \Delta y} (\mathbf{U}^p_{E,i} n-1 \mathbf{H}_{i,j}^f)$$

$$+ \mathbf{U}^p_{E,i} n-1 \mathbf{H}_{i,j-1}^f$$

(59)

This equation is the same as (51) except for the use of the \( \mathbf{U}_p \) matrices. Thus, it is possible to implement subcell PEC modeling in MRTD while changing only the inner product matrices. This method adds no increase in computational overhead, and simply requires the additional memory to store the \( \mathbf{U} \) matrices.

### 7.2.3. Examples

To test the method, a PEC screen in a 2D parallel-plate waveguide was simulated. An expanded view of the grid surrounding the screen is presented in Fig. 14. The areas where the PEC is applied are shaded. A maximum wavelet resolution of 2 was used. In this simulation, the voltage at the output of the screen was probed. When compared to the results of an FDTD simulation of the same structure (Fig. 15), the maximum difference in magnitude is shown on the order of 10–12% of the peak field value. This is within the numerical accuracy of Matlab [69], which was used for the simulations, showing that the techniques are identical.

#### 7.2.3.1. CPW–Microstrip Transition

The CPW–microstrip transition simulated is shown in Fig. 16. The loss of this transition can be optimized over a wide frequency range with the use of FDTD and design curves for various packaging specifications can be derived. The plot in Fig. 17 shows \( S_{21} \) of this transition for a variety of lengths of the...
central straight section from 10 to 20 GHz. This data was obtained using time-domain voltage probes at the input \((V_1)\) and output \((V_2)\) of the transition, converting them to frequency domain through the use of a discrete Fourier transform and identifying the reflected voltage through the use of a reference input voltage \((V_{\text{ref}})\) derived by the simulation of a through CPW line [64]. In addition, the use of the full-wave FDTD, which provides the values of all electromagnetic components throughout the geometry, offers a more intuitive visualization of the circuit. For example, in the transition the electric fields have to change smoothly from a coplanar waveguide mode to a microstrip mode, in order to minimize the local reflections. Thus, in the design process it is desirable to identify where this transition takes place and optimize the tapering. Figure 16 is a plot of total electric field for a transverse cross section of the transition. It can be seen that at the position of this cross section, the field is mostly in a CPW mode, although a microstrip mode has started developing below and at the edges of the signal line there. The relative amplitudes of the \(E\) field could provide an intuitive design rule for the spacing between the CPW ground and signal line, so as not to suppress the microstrip mode.

7.2.3.2. Microstrip-Line Coupling. Embedded transmission lines are commonly used in multilayer packages, where the use of noncontinuous grounds could lead to increased crosstalk effects. In this article, the FDTD technique is used for the estimation of the coupling of the finite-ground microstrip lines of Fig. 18 [66]. The results for different line spacing and for a ground connecting via (optimized design) presented in Fig. 19 have been obtained by combining two simulations, an even/odd-mode excitation. In addition, to reduce the unwanted crosstalk, the electric and the magnetic field distributions have been calculated and plotted along a transverse cross section. It is apparent that most of the coupling is through the magnetic field lines, leading to the design conclusion that attempts to reduce the coupling should focus on magnetic shielding.

7.2.3.3. MEMS Capacitive Switch. One example of a MEMS structure that benefits from simulation in MRTD [67] is the MEMS capacitive switch shown in Fig. 20. The gap between the plates in the switch is \(\frac{1}{175}\)th of the substrate thickness. The simulation of this device in FDTD is tedious and slow because of the large number of cells that must be used in order to accurately represent the very small gap and substrate.

In MRTD, the number of cells can be reduced by using the built-in adaptive gridding capability of the method. In addition, further efficiencies can be obtained in large
simulations featuring this structure by allowing fewer cells to be used when the electric field variation near the cell is low.

7.2.3.4. LTCC Antenna Development. A via-fed stacked cavity-backed patch antenna has been designed (based on a 10-layer LTCC process) for IEEE 802.11a 5.8 GHz band [68] as shown in Fig. 21. The heights of the lower and upper patches (400 mils, 400 mils) are respectively 8 mils (2 LTCC GL550 layers) and 32 mils.

The input impedance and radiation characteristics of the stacked-patch antenna are shown in Figs. 22 and 23 respectively. The 10-dB return loss bandwidth of the antenna is about 4.

7.2.4. Concluding Remarks. The multiresolution time domain (MRTD) is a wavelet-based extension of FDTD. It demonstrates a very high efficiency in the calculation of the scattering parameters as well as its the estimation of the packaging effects and of the parasitic crosstalk between neighboring geometries. In addition, its inherent capability of global electromagnetic field calculation as well as MRTD’s multi-PEC cell allows for the identification of “hotspots” of high field concentration and for the derivation of physically driven solutions for improvement of the overall system-on-package efficiency. The time/space-adaptive grid of MRTD allows it to be used to model finely detailed structures. Areas of the grid containing small features can use increased resolution, while homogenous areas can use low resolution. It is important to note that this technique can be used to model structures with continuous dielectric variations, and thus composite cells, that is, those with multiple PEC and dielectric regions per cell, can be modeled. Furthermore, the pointwise expansion of MRTD equations can be used in the future to model other subcell components, such as lumped elements and equivalent circuits.

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BIBLIOGRAPHY


